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THREE-DIMENSIONAL EIGENFUNCTION ANALYSIS OF A CRACK IN A PIEZOELECTRIC MATERIAL

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Abstract—A three-dimensional analysis of a semi-infinite crack embedded in a transversely isotropic piezoelectric material was performed by means of the eigenfunction expansions method. The results show the characteristic $1/\sqrt{r}$ singular behavior of the stress tensor in the vicinity of the crack. A similar behavior is also revealed for the electric field. In addition the study demonstrates that coupling (or decoupling) of the mechanical and electrical variables is a function of the distance to the crack tip as well as of the crack orientation.

INTRODUCTION

Due to the intrinsic coupling effects that take place between electric fields and mechanical deformation, piezoelectric materials have been extensively used as transducers and sensors. More recently, they are playing a key role as active components such as sonar projectors and pulse generators. Furthermore, cofired multilayer lead zirconate titanate (also known as PZT) stacks are under development for potential use in high drive sonars as well as ceramic actuators. In these new fields of application, severe mechanical stressing occurs during operation. For example, in the case of multilayer stacks, the electrodes that terminate inside the ceramic body are a source of electric field concentration which can result in stress concentrations high enough to fracture the parts. Reliable service lifetime predictions of piezoceramic components demand a complete understanding of the fracture processes of these materials. Because piezoelectric materials can deform under both applied mechanical and electrical loads, a study of the effects of electric fields on crack propagation is of both theoretical and practical interest. Despite the fact that piezoelectrics have been in use for decades in electromechanical devices, very little theoretical work has been done concerning their mechanical failure. To the authors' knowledge only the works of Parton (1976) and Deeg (1980) seem to have addressed the fracture problem in piezoelectric materials from a theoretical stand point. More recently, Pak (1987) obtained a closed form solution to an antiplane fracture problem. In his paper, the mode III fracture behavior of a finite crack embedded in an infinite piezoelectric solid was studied. It was shown that the mechanical and electrical fields exhibit the classical mode III singularity at the crack tip. It was also shown that crack growth can be enhanced or retarded depending on the magnitude and the direction of the applied electrical load. This observation agrees with that of Deeg (1980) who has considered an in-plane piezoelectric fracture problem using the distributed dislocation method. Although significant experimental work has been carried out to study the strength and toughness of piezoceramics containing defects (see Freiman, 1986; Harrison *et al.*, 1986; and Pisarenko *et al.*, 1985), in all cases the applied electric field has been considered an environmental effect. Thus at present no clear picture exists of the effects of an electric field on the fracture process from a continuum mechanics point of view. In an attempt to obtain a better understanding of the mechanics of fracture in the presence of mechanical and electrical loadings, this paper will focus on a three-dimensional study of a semi-infinite crack embedded in an unbounded transversely isotropic piezoelectric material. The analysis is carried out with the aid of the eigenfunction expansions method which was introduced by Hartranft and Sih (1969) in order to study the purely elastic three-dimensional crack problem. The present analysis makes it possible to obtain the singular behavior of the elastic and the electric fields near a crack for all three possible modes of fracture. In addition, for a given crack orientation it will be possible to examine the coupling (or decoupling) that occurs between the mechanical and the electrical fields according to their proximity to the crack tip.

GOVERNING EQUATIONS

Following Tiersten (1969) the equations governing the three-dimensional theory of piezoelectricity in the absence of body forces and free charges can be written in compact manner as follows:

$$\sigma_{ij,j} = 0 \tag{1}$$

$$D_{i,i} = 0 \tag{2}$$

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k \tag{3}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k \tag{4}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}) \tag{5}$$

$$E_i = -\phi_{,i}, \tag{6}$$

where $i, j, k, l = 1, 2, 3$, and σ_{ij}, D_i, ε_{ij}, u_i, E_i and ϕ are the components of stress, electric displacement, strain, displacement, electric field and electric potential, respectively. Equations (3) and (4) represent the piezoelectric stress constitutive relationships and are expressed in terms of the elastic stiffness constants c_{ijkl} (measured in a constant electric field), the dielectric constants ϵ_{ij} (measured at constant strain) and the piezoelectric constants e_{ijk}. In the most general case of anisotropy (triclinic crystal structure), the piezoelectric material is described by $21+6+18 = 45$ independent constants.

The present study is concerned with a transversely isotropic (or unidirectionally anisotropic) piezoelectric, that is, a material with the symmetry of a hexagonal crystal class 6 *mm*. In Cartesian coordinates (with the z-axis being positioned normal to the plane of isotropy), the constitutive equations reduce to expressions (A1) and (A2) of the Appendix, from which it can be deduced that the solid is characterized by five elastic, two dielectric and three piezoelectric constants, that is, a total of 10 independent material constants. It is to be noted that materials with this type of symmetry (for example PZT ceramics) possess high piezoelectric coupling. Consequently, the proposed analysis constitutes a relevant as well as an interesting approach towards understanding the effects that electric fields have on the propagation of cracks contained in piezoelectric ceramics.

THE CRACK PROBLEM

In this section we are concerned with the analysis of the stress and electric displacement fields (or induction) in the vicinity of a semi-infinite plane crack embedded in an infinite transversely isotropic piezoelectric medium.

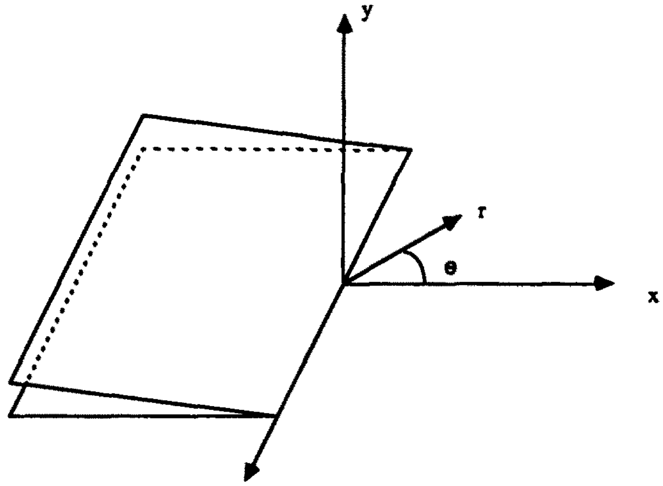

Fig. 1. Coordinates used in the crack analysis.

The problem becomes more tractable if we introduce a system of cylindrical coordinates (r, θ, z) as illustrated in Fig. 1. Presently, we are interested in a particular crack orientation, namely one in which the crack front is assumed to be straight and located along the z-axis, while the y-axis is perpendicular to the crack planes. Such a configuration will render a particular interaction between the stresses and the electric field as will be shown. The study of other crack orientations and their consequences will be the subject of a forthcoming paper.

According to Fig. 1, the limits of the cylindrical coordinates are

$$0 \leqslant r < \infty, \quad -\pi \leqslant \theta \leqslant \pi, \quad -\infty < z < \infty.$$

With respect to this new coordinate system the equations of equilibrium for the stress and the electric displacement components are given by

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{(\sigma_{rr}-\sigma_{\theta\theta})}{r}=0 \\ &\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{\theta z}}{\partial z}+2\frac{\sigma_{r\theta}}{r}=0 \\ &\frac{\partial \sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=0 \\ &\frac{\partial D_r}{\partial r}+\frac{1}{r}\frac{\partial D_\theta}{\partial \theta}+\frac{\partial D_z}{\partial z}+\frac{D_r}{r}=0, \end{aligned} \tag{7}$$

while the constitutive equations are still given by eqns (A1) and (A2), but with the stress, strain, electric field and electric induction matrices now transformed to cylindrical coordinates. These relationships can be rewritten in terms of the elastic displacement and electric potential gradients in the following manner:

$$\sigma_{rr}=c_{11}\frac{\partial u_r}{\partial r}+c_{12}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+c_{13}\frac{\partial u_z}{\partial z}+e_{31}\frac{\partial \phi}{\partial z}$$

$$\sigma_{\theta\theta}=c_{12}\frac{\partial u_r}{\partial r}+c_{11}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+c_{13}\frac{\partial u_z}{\partial z}+e_{31}\frac{\partial \phi}{\partial z}$$

$$\sigma_{zz} = c_{13}\frac{\partial u_r}{\partial r} + c_{13}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + c_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \phi}{\partial z}$$

$$\sigma_{z\theta} = c_{44}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) + e_{15}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\sigma_{zr} = c_{44}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + e_{15}\frac{\partial \phi}{\partial r}$$

$$\sigma_{r\theta} = \tfrac{1}{2}(c_{11} - c_{12})\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \tag{8}$$

$$D_r = e_{15}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) - \epsilon_{11}\frac{\partial \phi}{\partial r}$$

$$D_\theta = e_{15}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) - \epsilon_{11}\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$D_z = e_{31}\frac{\partial u_r}{\partial r} + e_{31}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + e_{33}\frac{\partial u_z}{\partial z} - \epsilon_{33}\frac{\partial \phi}{\partial z}, \tag{9}$$

where (u_r, u_θ, u_z) are the cylindrical components of the displacement vector.

As was shown by Hartranft and Sih (1969), the three-dimensional elastic crack problem can be solved in a systematic manner by representing the displacement vector in terms of a double infinite series of arbitrary functions of the cylindrical coordinates. In the present study, the same idea is introduced and extended in order to include the behavior of the electric field as well. Therefore, it is suitable to write

$$(u_r, u_\theta, u_z, \phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^{\lambda_m + n}[R_n^{(m)}, \Theta_n^{(m)}, Z_n^{(m)}, \Phi_n^{(m)}], \tag{10}$$

where the functions $R_n^{(m)}$, $\Theta_n^{(m)}$, $Z_n^{(m)}$, and $\Phi_n^{(m)}$ depend on θ, z and λ_m, and are equal to zero for $n < 0$ in order to avoid unbounded displacements and electric potential in the neighborhood of the crack tip. The eigenvalues λ_m $(m = 0, 1, \ldots)$ as powers of r, are assumed to be constants, yet to be determined. When the above functions are substituted into the constitutive equations, we obtain

$$\sigma_{rr} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^{\lambda_m + n - 1}\left[(\lambda_m + n)c_{11}R_n^{(m)} + c_{12}\left(\frac{\partial \Theta_n^{(m)}}{\partial \theta} + R_n^{(m)}\right) + c_{13}\frac{\partial Z_{n-1}^{(m)}}{\partial z} + e_{31}\frac{\partial \Phi_{n-1}^{(m)}}{\partial z}\right]$$

$$\vdots \tag{11}$$

$$D_z = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^{\lambda_m + n - 1}\left[(\lambda_m + n)e_{31}R_n^{(m)} + e_{31}\left(\frac{\partial \Theta_n^{(m)}}{\partial \theta} + R_n^{(m)}\right) + e_{33}\frac{\partial Z_{n-1}^{(m)}}{\partial z} - \epsilon_{33}\frac{\partial \Phi_{n-1}^{(m)}}{\partial z}\right],$$

where for brevity we show in full only two of the equations shown in (8) and (9). Next, the stresses and the electric displacements given by eqn (11) are substituted into eqn (7), leading to expressions that will hold for arbitrary powers of r. As a consequence, a system of four coupled partial differential equations is obtained, namely:

$$\tfrac{1}{2}(c_{11} - c_{12})\frac{\partial^2 R_n^{(m)}}{\partial \theta^2} + [(\lambda_m + n)^2 - 1]c_{11}R_n^{(m)} + \tfrac{1}{2}[(\lambda_m + n - 3)c_{11} + (\lambda_m + n + 1)c_{12}]\frac{\partial \Theta_n^{(m)}}{\partial \theta}$$

$$= -(\lambda_m + n - 1)\left[(c_{13} + c_{44})\frac{\partial Z_{n-1}^{(m)}}{\partial z} + (e_{31} + e_{15})\frac{\partial \Phi_{n-1}^{(m)}}{\partial z}\right] + c_{44}\frac{\partial^2 R_{n-2}^{(m)}}{\partial z^2}, \tag{12.1}$$

$$c_{11}\frac{\partial^2\Theta_n^{(m)}}{\partial\theta^2}+\tfrac{1}{2}[(\lambda_m+n+3)c_{11}+(\lambda_{m+n-1})c_{12}]\frac{\partial R_n^{(m)}}{\partial\theta}+\tfrac{1}{2}[(\lambda_m+n)^2-1](c_{11}-c_{12})\Theta_n^{(m)}$$
$$= -(c_{13}+c_{44})\frac{\partial^2 Z_{n-1}^{(m)}}{\partial z\,\partial\theta}-(e_{31}+e_{15})\frac{\partial^2\Phi_{n-1}^{(m)}}{\partial z\,\partial\theta}-c_{44}\frac{\partial^2\Theta_{n-2}^{(m)}}{\partial z^2}, \quad (12.2)$$

$$c_{44}\frac{\partial^2 Z_n^{(m)}}{\partial\theta^2}+(\lambda_m+n)^2c_{44}Z_n^{(m)}+e_{15}\frac{\partial^2\Phi_n^{(m)}}{\partial\theta^2}+(\lambda_m+n)^2e_{15}\Phi_n^{(m)}$$
$$= -(c_{13}+c_{44})\frac{\partial^2\Theta_{n-1}^{(m)}}{\partial z\,\partial\theta}-(\lambda_m+n)(c_{13}+c_{44})\frac{\partial R_{n-1}^{(m)}}{\partial z}-c_{33}\frac{\partial^2 Z_{n-2}^{(m)}}{\partial z^2}-e_{33}\frac{\partial^2\Phi_{n-2}^{(m)}}{\partial z^2}, \quad (12.3)$$

$$-\epsilon_{11}\frac{\partial^2\Phi_n^{(m)}}{\partial\theta^2}-(\lambda_m+n)^2\epsilon_{11}\Phi_n^{(m)}+e_{15}\left[\frac{\partial^2 Z_n^{(m)}}{\partial\theta^2}+(\lambda_m+n)^2\right]Z_n^{(m)}$$
$$= -[(\lambda_m+n)(e_{15}+e_{31})]\frac{\partial R_{n-1}^{(m)}}{\partial z}-(e_{15}+e_{31})\frac{\partial^2\Theta_{n-1}^{(m)}}{\partial\theta\,\partial z}-c_{33}\frac{\partial^2 Z_{n-2}^{(m)}}{\partial z^2}-\epsilon_{33}\frac{\partial^2\Phi_{n-2}^{(m)}}{\partial z^2}. \quad (12.4)$$

The above system can now be solved for different values of n, where solutions for $R_n^{(m)}$, $\Theta_n^{(m)}$, $Z_n^{(m)}$ and $\Phi_n^{(m)}$ in terms of θ and z will be obtained in terms of their previous values. The first step towards achieving the crack solution is to evaluate the eigenvalues λ_m. Since the eigenvalues are independent of n, without loss in generality, we can simplify the problem by determining the eigenvalues for the case of $n = 0$. For such a particular case the four right-hand sides of eqns (12.1)–(12.4) vanish, and the original set of equations will then decouple into two systems independent of each other—one system of equations containing the functions $R_0^{(m)}$ and $\theta_0^{(m)}$, while the other coupling $Z_0^{(m)}$ with $\Phi_0^{(m)}$, as is shown below.

$$\tfrac{1}{2}(c_{11}-c_{12})\frac{\partial^2 R_0^{(m)}}{\partial\theta^2}+(\lambda_m^2-1)c_{11}R_0^{(m)}+\tfrac{1}{2}[(\lambda_m-3)c_{11}+(\lambda_m+1)c_{12}]\frac{\partial\Theta_0^{(m)}}{\partial\theta}=0$$
$$c_{11}\frac{\partial^2\Theta_0^{(m)}}{\partial\theta^2}+\tfrac{1}{2}(\lambda_m^2-1)(c_{11}-c_{12})\Theta_0^{(m)}+\tfrac{1}{2}[(\lambda_m+3)e_{11}+(\lambda_m-1)c_{12}]\frac{\partial R_0^{(m)}}{\partial\theta}=0, \quad (13.1)$$

$$c_{44}\left[\frac{\partial^2 Z_0^{(m)}}{\partial\theta^2}+\lambda_m^2 Z_0^{(m)}\right]+e_{15}\left[\frac{\partial^2\Phi_0^{(m)}}{\partial\theta^2}+\lambda_m^2\Phi_0^{(m)}\right]=0$$
$$-\epsilon_{11}\left[\frac{\partial^2\Phi_0^{(m)}}{\partial\theta^2}+\lambda_m^2\Phi_0^{(m)}\right]+e_{15}\left[\frac{\partial^2 Z_0^{(m)}}{\partial\theta^2}+\lambda_m^2 Z_0^{(m)}\right]=0. \quad (13.2)$$

The system given by eqn (13.2) has a straightforward solution in terms of the product of functions of z and θ, that is

$$Z_0^{(m)} = A_0(z)\cos\lambda_m\theta+B_0(z)\sin\lambda_m\theta$$
$$\Phi_0^{(m)} = C_0(z)\cos\lambda_m\theta+D_0(z)\sin\lambda_m\theta, \quad (14)$$

where $A_0(z)$, $B_0(z)$, $C_0(z)$, $D_0(z)$ are arbitrary functions of z. In order to find the solution to eqn (13.1) we reduce the system to two independent fourth-order ordinary differential equations of the form

$$\Delta R_0^{(m)} = 0, \quad \Delta\Theta_0^{(m)} = 0, \quad (15)$$

with Δ given by

$$\Delta = L_1 L_4 - L_2 L_3, \tag{16}$$

where L_1, L_2, L_3, L_4 are differential operators applied to the functions $R_0^{(m)}$ and $V_0^{(m)}$; for example

$$L_1 = \tfrac{1}{2}(c_{11} - c_{12})D^2 + (\lambda_m^2 - 1),$$

and similar expressions for the other three operators. Here we have used the notation $D^2 = \partial^2/\partial\theta^2$ to denote second-order differentiation. Equation (16) will give rise to a fourth-order algebraic equation, which once set equal to zero produces the roots

$$r = \pm(\lambda_m + 1)i, \quad \pm(\lambda_m - 1)i. \tag{17}$$

Consequently, the solutions to eqn (15) are given by

$$R_0^{(m)} = E_0(z)\cos(\lambda_m + 1)\theta + F_0(z)\sin(\lambda_m + 1)\theta + G_0(z)\cos(\lambda_m - 1)\theta + H_0(z)\sin(\lambda_m - 1)\theta$$
$$\Theta_0^{(m)} = I_0(z)\cos(\lambda_m + 1)\theta + J_0(z)\sin(\lambda_m + 1)\theta + K_0(z)\cos(\lambda_m - 1)\theta + L_0(z)\sin(\lambda_m - 1)\theta, \tag{18}$$

where the functions $E_0(z), \ldots, L_0(z)$ are yet to be determined. However, not all of these arbitrary functions are independent. In fact, relationships among them can be established by substituting the above solutions into eqn (15). For example $J_0(z)$ can be written in terms of $E_0(z)$ as follows:

$$J_0(z) = (\lambda_m + 1)\frac{(\lambda_m - 3)c_{11} - (\lambda_m + 1)c_{12}}{(\lambda_m - 3)c_{11} + (\lambda_m + 1)c_{12}}E_0(z), \tag{19}$$

with similar relationships among the other six functions, which for the sake of brevity are omitted. In order to find the eigenvalues λ_m, the next step is to make use of the crack surface boundary conditions. This implies the substitution of eqns (14) and (18) into eqn (11) when $n = 0$. It is assumed that the crack faces are free of surface traction and surface charge and that the crack is filled with vacuum. Under these conditions it is quite appropriate to write (Pak, 1987):

$$\sigma_{\theta\theta} = \sigma_{\theta r} = \sigma_{\theta z} = D_\theta = 0, \quad \text{at} \quad \theta = \pm\pi. \tag{20}$$

As a result, a system of eight algebraic equations in the eight unknowns $A_0(z), \ldots, H_0(z)$ is obtained. A non-trivial solution of the system is obtained if the 8×8 determinant of the coefficients is equal to zero. However, due to the particular coupling expressed by eqns (13.1) and (13.2), we can solve two 4×4 determinants independent of each other, which both lead to the same eigen-equation, namely

$$\sin 2\lambda_m \pi = 0, \tag{21}$$

which renders the eigenvalues

$$\lambda_m = \frac{m}{2}, \quad m = 0, 1, 2, \ldots, \tag{22}$$

where negative values of m are excluded in order to insure bounded displacements and electric potential at the crack tip. We conclude from (22) that r varies as r^0, $r^{1/2}$, r^1, $r^{3/2}$, etc.

Once the functional form of the radial variable is known, the field variables can be expressed in terms of a single series. To this end we introduce new functions f_n, g_n, h_n, φ_n depending on θ and z only. Therefore, we can write

$$(u_r, u_\theta, u_z, \phi) = \sum_{n=0}^{\infty} r^{n/2}[f_n(\theta,z), g_n(\theta,z), h_n(\theta,z), \varphi_n(\theta,z)]. \tag{23}$$

We now proceed in the same manner that led to eqn (12), that is, we substitute eqn (23) into eqns (8) and (9) which yields

$$\begin{aligned}
\sigma_{rr} &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\left(\frac{n}{2}c_{11}+c_{12}\right)f_n + c_{12}\frac{\partial g_n}{\partial\theta} + c_{13}\frac{\partial h_{n-2}}{\partial z} + e_{31}\frac{\partial\varphi_{n-2}}{\partial z}\right] \\
\sigma_{\theta\theta} &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\left(\frac{n}{2}c_{12}+c_{11}\right)f_n + c_{11}\frac{\partial g_n}{\partial\theta} + c_{13}\frac{\partial h_{n-2}}{\partial z} + e_{31}\frac{\partial\varphi_{n-2}}{\partial z}\right] \\
\sigma_{zz} &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\left(\frac{n}{2}+1\right)c_{13}f_n + c_{13}\frac{\partial g_n}{\partial\theta} + c_{33}\frac{\partial h_{n-2}}{\partial z} + e_{33}\frac{\partial\varphi_{n-2}}{\partial z}\right] \\
\sigma_{z\theta} &= \sum_{n=0}^{\infty} r^{n/2-1}\left[c_{44}\frac{\partial h_n}{\partial\theta} + e_{15}\frac{\partial\varphi_n}{\partial\theta} + c_{44}\frac{\partial g_{n-2}}{\partial z}\right] \\
\sigma_{zr} &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\frac{n}{2}c_{44}h_n + \frac{n}{2}e_{15}\varphi_n + c_{44}\frac{\partial f_{n-2}}{\partial z}\right] \\
\sigma_{r\theta} &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\frac{1}{2}(c_{11}-c_{12})\left[\frac{\partial f_n}{\partial\theta} + \left(\frac{n}{2}-1\right)g_n\right]\right],
\end{aligned} \tag{24}$$

$$\begin{aligned}
D_r &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\frac{n}{2}e_{15}h_n - \frac{n}{2}\epsilon_{11}\varphi_n + e_{15}\frac{\partial f_{n-2}}{\partial z}\right] \\
D_\theta &= \sum_{n=0}^{\infty} r^{n/2-1}\left[e_{15}\frac{\partial h_n}{\partial\theta} - \epsilon_{11}\frac{\partial\varphi_n}{\partial\theta} + e_{15}\frac{\partial g_{n-2}}{\partial z}\right] \\
D_z &= \sum_{n=0}^{\infty} r^{n/2-1}\left[\left(\frac{n}{2}+1\right)e_{31}f_n + e_{31}\frac{\partial g_n}{\partial\theta} + e_{33}\frac{\partial h_{n-2}}{\partial z} - \epsilon_{33}\frac{\partial\varphi_{n-2}}{\partial z}\right].
\end{aligned} \tag{25}$$

Further substitution of the above relations into eqn (7) generates a new system of four partial differential equations which can be viewed as recurrence relations for the functions f_n, g_n, h_n, φ_n. These equations are:

$$\begin{aligned}
&\tfrac{1}{2}(c_{11}-c_{12})\frac{\partial^2 f_n}{\partial\theta^2} + \left(\frac{n^2}{4}-1\right)c_{11}f_n + \frac{1}{2}\left[\left(\frac{n}{2}-3\right)c_{11} + \left(\frac{n}{2}+1\right)c_{12}\right]\frac{\partial g_n}{\partial\theta} \\
&\quad = -\left(\frac{n}{2}-1\right)(c_{13}+c_{44})\frac{\partial h_{n-2}}{\partial z} - \left(\frac{n}{2}-1\right)(e_{31}+e_{15})\frac{\partial\varphi_{n-2}}{\partial z} - c_{44}\frac{\partial^2 f_{n-4}}{\partial z^2},
\end{aligned} \tag{26.1}$$

$$\begin{aligned}
&c_{11}\frac{\partial^2 g_n}{\partial\theta^2} + \frac{1}{2}\left(\frac{n^2}{4}-1\right)(c_{11}-c_{12})g_n + \frac{1}{2}\left[\left(\frac{n}{2}+3\right)c_{11} + \left(\frac{n}{2}-1\right)c_{12}\right]\frac{\partial f_n}{\partial\theta} \\
&\quad = -(c_{13}+c_{44})\frac{\partial^2 h_{n-2}}{\partial z\,\partial\theta} - (e_{31}+e_{15})\frac{\partial^2\varphi_{n-2}}{\partial z\,\partial\theta} - c_{44}\frac{\partial^2 g_{n-4}}{\partial z^2},
\end{aligned} \tag{26.2}$$

$$c_{44}\left[\frac{\partial^2 h_n}{\partial\theta^2}+\frac{n^2}{4}h_n\right]+e_{15}\left[\frac{\partial^2\varphi_n}{\partial\theta^2}+\frac{n^2}{4}\varphi_n\right]=-\frac{n}{2}(c_{13}+c_{44})\frac{\partial f_{n-2}}{\partial z}$$
$$-(c_{13}+c_{44})\frac{\partial^2 g_{n-2}}{\partial z\,\partial\theta}-c_{33}\frac{\partial^2 h_{n-4}}{\partial z^2}-c_{33}\frac{\partial^2\varphi_{n-4}}{\partial z^2}, \quad (26.3)$$

$$-\epsilon_{11}\left[\frac{\partial^2\varphi_n}{\partial\theta^2}+\frac{n^2}{4}\varphi_n\right]+e_{15}\left[\frac{\partial^2 h_n}{\partial\theta^2}+\frac{n^2}{4}h_n\right]=-\frac{n}{2}(e_{31}+e_{15})\frac{\partial f_{n-2}}{\partial z}$$
$$-(e_{31}+e_{15})\frac{\partial^2 g_{n-2}}{\partial z\,\partial\theta}-e_{33}\frac{\partial^2 h_{n-4}}{\partial z^2}+\epsilon_{33}\frac{\partial^2\varphi_{n-4}}{\partial z^2}. \quad (26.4)$$

The solution of the above set of equations for all possible values of n provides the behavior of the stress and electric field components anywhere in the piezoelectric medium. In the most general case the four equations will be coupled and particular solutions will need to be found. However, as observed by inspection, the equations become homogeneous for the values of $n = 0$ and $n = 1$. Less obvious, but still true as will be shown, is the fact that for $n = 2$ the four right-hand sides also vanish. In addition, for these particular values of n a substantial simplification of the problem is achieved due to the fact that eqns (26.1) and (26.2) will couple f_n and g_n only, while eqns (26.3) and (26.4) will relate the functions h_n and φ_n.

THE SOLUTION FOR THE LEADING TERMS

Since our main concern from a fracture mechanics point of view is the behavior of the fields in the vicinity of the crack tip, we will restrict ourselves to the solution of eqns (26.1)–(26.4) for only the first three values of n. Towards this end we first analyze eqns (26.1)–(26.2) which as before can be reduced to two fourth-order ordinary differential equations for f_n and g_n. Thus we can write

$$\Delta f_n = 0, \quad \Delta g_n = 0, \quad (27)$$

with Δ given again by eqn (16), but with the differential operators L_i ($i = 1, 2, 3, 4$) expressed in the following form:

$$L_1 = \tfrac{1}{2}(c_{11}-c_{12})D^2+\left(\frac{n^2}{4}-1\right)c_{11}$$
$$L_2 = \frac{1}{2}\left[\left(\frac{n}{2}-3\right)c_{11}+\left(\frac{n}{2}+1\right)c_{12}\right]D$$
$$L_3 = \frac{1}{2}\left[\left(\frac{n}{2}+3\right)c_{11}+\left(\frac{n}{2}-1\right)c_{12}\right]D$$
$$L_4 = c_{11}D^2+\frac{1}{2}\left(\frac{n^2}{4}-1\right)(c_{11}-c_{12}), \quad (28)$$

where D and D^2 stand for $\partial/\partial\theta$, and $\partial^2/\partial\theta^2$, respectively. The expansion of eqn (16) using (28) leads to

$$\Delta = \tfrac{1}{2}(c_{11}-c_{12})c_{11}D^4+\left(\frac{n^2}{4}+1\right)(c_{11}-c_{12})c_{11}D^2+\frac{1}{2}\left(\frac{n^2}{4}-1\right)(c_{11}-c_{12})c_{11}. \quad (29)$$

Replacing D by r and equating to zero we obtain the fourth-degree characteristic equation

$$r^4+2\left(\frac{n^2}{4}+1\right)r^2+\left(\frac{n^2}{4}-1\right)=0, \tag{30}$$

with roots

$$r=\pm\left(\frac{n}{2}+1\right)i,\quad \pm\left(\frac{n}{2}-1\right)i. \tag{31}$$

Thus it follows that the solutions to (27) can be expressed as

$$\begin{aligned} f_n(\theta,z) &= F_n^{(1)}(z)\cos\left(\frac{n}{2}+1\right)\theta+F_n^{(2)}(z)\sin\left(\frac{n}{2}+1\right)\theta \\ &\quad +F_n^{(3)}(z)\cos\left(\frac{n}{2}-1\right)\theta+F_n^{(4)}(z)\sin\left(\frac{n}{2}-1\right)\theta \\ g_n(\theta,z) &= G_n^{(1)}(z)\cos\left(\frac{n}{2}+1\right)\theta+G_n^{(2)}(z)\sin\left(\frac{n}{2}+1\right)\theta \\ &\quad +G_n^{(3)}(z)\cos\left(\frac{n}{2}-1\right)\theta+G_n^{(4)}(z)\sin\left(\frac{n}{2}-1\right)\theta, \end{aligned} \tag{32}$$

where the coefficients $F_n^{(1)}(z),\ldots,G_n^{(4)}(z)$ constitute eight independent arbitrary functions of z. Although it is true that solutions to the original system formed by (26.1)–(26.2) are contained in the solutions to eqn (27), the converse is not generally true. In fact, eqn (32) will be a solution to the original system of equations if certain relationships exist among the eight functions $F_n^{(i)}(z), G_n^{(i)}(z)$ $(i=1,2,3,4)$. These relationships are obtained by substituting eqn (32) into the first two of (26), leading to the following results:

$$G_n^{(1)}(z)=F_n^{(2)}(z),\quad G_n^{(2)}(z)=-F_n^{(1)}(z),\quad G_n^{(3)}(z)=\alpha_n F_n^{(4)}(z),\quad G_n^{(4)}(z)=-\alpha_n F_n^{(3)}(z), \tag{33}$$

where

$$\alpha_n=\frac{(n/2+3)c_{11}+(n/2-1)c_{12}}{(n/2-3)c_{11}+(n/2+1)c_{12}}. \tag{34}$$

Now the solutions (32) can be written as

$$\begin{aligned} f_n(\theta,z) &= A_n^{(1)}(z)\cos\left(\frac{n}{2}+1\right)\theta+A_n^{(2)}(z)\sin\left(\frac{n}{2}+1\right)\theta \\ &\quad +B_n^{(1)}(z)\cos\left(\frac{n}{2}-1\right)\theta+B_n^{(2)}(z)\sin\left(\frac{n}{2}-1\right)\theta \\ g_n(\theta,z) &= A_n^{(2)}(z)\cos\left(\frac{n}{2}+1\right)\theta-A_n^{(1)}(z)\sin\left(\frac{n}{2}+1\right)\theta \\ &\quad +\alpha_n\left[B_n^{(2)}(z)\cos\left(\frac{n}{2}-1\right)\theta-B_n^{(1)}(z)\sin\left(\frac{n}{2}-1\right)\theta\right], \end{aligned} \tag{35}$$

where we note the renaming of the functions $F_n^{(i)}(z)$.

Turning now our attention to the system of equations formed by (26.3) and (26.4), after simple algebraic manipulation we can write

$$(c_{44}\epsilon_{11}+e_{15}^2)\left[\frac{\partial^2 h_n}{\partial\theta^2}+\frac{n^2}{4}h_n\right]=0$$

$$(e_{15}^2+\epsilon_{11}e_{15})\left[\frac{\partial^2 \varphi_n}{\partial\theta^2}+\frac{n^2}{4}\varphi_n\right]=0, \tag{36}$$

which have as solutions

$$h_n(\theta,z)=C_n^{(1)}(z)\cos\frac{n}{2}\theta+C_n^{(2)}(z)\sin\frac{n}{2}\theta$$

$$\varphi_n(\theta,z)=D_n^{(1)}(z)\cos\frac{n}{2}\theta+D_n^{(2)}(z)\sin\frac{n}{2}\theta. \tag{37}$$

The next step is the determination of some of the independent coefficient functions $A_n^{(i)}(z)$, $B_n^{(i)}(z)$, $C_n^{(i)}(z)$, $D_n^{(i)}(z)$ $(i=1,2)$ by introducing the crack face boundary conditions (20), which in terms of the eigenfunctions take the form

$$\frac{\partial f_n}{\partial\theta}+\left(\frac{n}{2}-1\right)g_n=0$$

$$\left(\frac{n}{2}c_{12}+c_{11}\right)f_n+c_{11}\frac{\partial g_n}{\partial\theta}+c_{13}\frac{\partial h_{n-2}}{\partial z}+e_{31}\frac{\partial\varphi_{n-2}}{\partial z}=0$$

$$c_{44}=\frac{\partial h_n}{\partial\theta}+e_{15}\frac{\partial\varphi_n}{\partial\theta}+c_{44}\frac{\partial g_{n-2}}{\partial z}=0$$

$$e_{15}\frac{\partial h_n}{\partial\theta}-\epsilon_{11}\frac{\partial\varphi_n}{\partial\theta}+e_{15}\frac{\partial g_{n-2}}{\partial z}=0, \tag{38}$$

being evaluated at $\theta=\pm\pi$.

A summary of the results obtained for the first three values of n, with the corresponding expressions for the stress and electric induction components, are presented below.

(i) *Case* n = 0

By means of eqns (35), (37) and (38) we obtain

$$B_0^{(1)}=B_0^{(2)}=C_0^{(2)}=D_0^{(2)}=0, \tag{39}$$

which allows us to write

$$f_0=A_0^{(1)}(z)\cos\theta+A_0^{(2)}(z)\sin\theta$$
$$g_0=A_0^{(2)}(z)\cos\theta-A_0^{(1)}(z)\sin\theta$$
$$h_0=C_0^{(1)}(z)$$
$$\varphi_0=D_0^{(1)}(z). \tag{40}$$

In order to describe the field variables anywhere in the piezoelectric solid we introduce the notation

$$\sigma_{rr} = \sum_{n=0}^{\infty} (\sigma_{rr})_n = (\sigma_{rr})_0 + (\sigma_{rr})_1 + \cdots, \tag{41}$$

with equivalent definitions for the rest of the stress and electric displacement components. Using (24), (25) and (40) we find that

$$(\sigma_{rr})_0 = (\sigma_{\theta\theta})_0 = (\sigma_{zz})_0 = (\sigma_{z\theta})_0 = (\sigma_{zr})_0 = (\sigma_{r\theta})_0 = 0, \tag{42}$$

$$(D_r)_0 = (D_\theta)_0 = (D_z)_0 = 0. \tag{43}$$

(ii) *Case* n = 1

In this case the boundary conditions yield

$$A_1^{(1)} = -\frac{c_{11}+c_{12}}{5c_{11}-3c_{12}} B_1^{(1)}, \quad A_1^{(2)} = \frac{3(c_{11}+c_{12})}{5c_{11}-3c_{12}} B_1^{(2)}, \quad D_1^{(1)} = C_1^{(1)} = 0, \tag{44}$$

and the eigenfunctions take the form

$$\begin{aligned}
f_1 &= A_1^{(1)}(z)\left[\cos\frac{3\theta}{2} - \frac{5c_{11}-3c_{12}}{c_{11}+c_{12}}\cos\frac{\theta}{2}\right] + A_1^{(2)}(z)\left[\sin\frac{3\theta}{2} - \frac{1}{3}\frac{5c_{11}-3c_{12}}{c_{11}+c_{12}}\sin\frac{\theta}{2}\right] \\
g_1 &= -A_1^{(1)}(z)\left[\sin\frac{3\theta}{2} - \frac{7c_{11}-c_{12}}{c_{11}+c_{12}}\sin\frac{\theta}{2}\right] + A_1^{(2)}(z)\left[\cos\frac{3\theta}{2} - \frac{1}{3}\frac{7c_{11}-c_{12}}{c_{11}+c_{12}}\cos\frac{\theta}{2}\right] \\
h_1 &= C_1^{(2)}(z)\sin\frac{\theta}{2} \\
\varphi_1 &= D_1^{(2)}(z)\sin\frac{\theta}{2},
\end{aligned} \tag{45}$$

which after algebraic manipulation leads to

$$\begin{aligned}
(\sigma_{rr})_1 &= \frac{1}{2\sqrt{r}}(c_{11}-c_{12})\left[A_1^{(1)}\left(\cos\frac{3\theta}{2} - 5\cos\frac{\theta}{2}\right) + A_1^{(2)}\left(\sin\frac{3\theta}{2} - \frac{5}{3}\sin\frac{\theta}{2}\right)\right] \\
(\sigma_{\theta\theta})_1 &= -\frac{1}{2\sqrt{r}}(c_{11}-c_{12})\left[A_1^{(1)}\left(\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}\right) + A_1^{(2)}\left(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right)\right] \\
(\sigma_{zz})_1 &= -\frac{4}{2\sqrt{r}}c_{13}\frac{(c_{11}-c_{12})}{c_{11}+c_{12}}\left[A_1^{(1)}\cos\frac{\theta}{2} + \frac{1}{3}A_1^{(2)}\sin\frac{\theta}{2}\right] \\
(\sigma_{z\theta})_1 &= \frac{1}{2\sqrt{r}}\left[c_{44}C_1^{(2)}\cos\frac{\theta}{2} + e_{15}D_1^{(2)}\cos\frac{\theta}{2}\right] \\
(\sigma_{zr})_1 &= \frac{1}{2\sqrt{r}}\left[c_{44}C_1^{(2)}\sin\frac{\theta}{2} + e_{15}D_1^{(2)}\sin\frac{\theta}{2}\right] \\
(\sigma_{r\theta})_1 &= \frac{1}{2\sqrt{r}}(c_{11}-c_{12})\left[-A_1^{(1)}\left(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}\right) + A_1^{(2)}\left(\cos\frac{3\theta}{2} + \frac{1}{3}\cos\frac{\theta}{2}\right)\right],
\end{aligned} \tag{46}$$

$$\begin{aligned}
(D_r)_1 &= \frac{1}{2\sqrt{r}}\left[e_{15}C_1^{(2)}\sin\frac{\theta}{2} - \epsilon_{11}D_1^{(2)}\sin\frac{\theta}{2}\right] \\
(D_\theta)_1 &= \frac{1}{2\sqrt{r}}\left[e_{15}C_1^{(2)}\cos\frac{\theta}{2} - \epsilon_{11}D_1^{(2)}\cos\frac{\theta}{2}\right]
\end{aligned}$$

$$(D_z)_1 = -4\frac{1}{\sqrt{r}}\frac{(c_{11}-c_{12})}{c_{11}+c_{12}}\left[A_1^{(1)}\cos\frac{\theta}{2}+\frac{1}{3}A_2^{(2)}\sin\frac{\theta}{2}\right]. \tag{47}$$

The preceding equations represent an important outcome of this article since they reveal the characteristic $1/\sqrt{r}$ type of singularity for both the stress and electric induction components. It is also worthwhile to note that in the case of a vanishing electric field, the angular variations of the stresses coincide with those of the purely elastic case considered by Hartranft and Sih (1969), although a substantial and expected difference resides in the material parameters involved in eqns (46).

(iii) *Case* n = 2

First we verify that the right-hand sides of eqns (26.1)–(26.4) indeed remain equal to zero by making use of eqn (40). Furthermore, the use of the boundary conditions now yields

$$A_2^{(2)} = 0, \quad C_2^{(2)} = -A_0^{(2)\prime}, \quad D_2^{(2)} = 0,$$
$$B_2^{(1)} = \frac{1}{c_{11}+c_{12}}[(c_{11}-c_{12})A_2^{(1)} - c_{13}C_0^{(1)\prime} - e_{31}D_0^{(1)\prime}], \tag{48}$$

where the primes denote derivatives with respect to z, and the solutions become

$$\begin{aligned}
f_2 &= A_2^{(1)}(z)\cos 2\theta + \frac{1}{c_{11}+c_{12}}[(c_{11}-c_{12})A_2^{(1)} - c_{13}C_0^{(1)\prime} - e_{31}D_0^{(1)\prime}]\\
g_2 &= -A_2^{(1)}(z)\sin 2\theta + B_2^{(2)}\\
h_2 &= C_2^{(1)}(z)\cos\theta - A_0^{(2)\prime}(z)\sin\theta\\
\varphi_2 &= D_2^{(1)}(z)\cos\theta,
\end{aligned} \tag{49}$$

which once substituted into (24) and (25) give the components of the field variables which are independent of the radial distance, namely

$$\begin{aligned}
(\sigma_{rr})_2 &= (c_{11}-c_{12})A_2^{(1)}(1+\cos 2\theta) - e_{31}D_0^{(1)\prime}\\
(\sigma_{\theta\theta})_2 &= (c_{11}-c_{12})A_2^{(1)}(1-\cos 2\theta) - e_{31}D_0^{(1)\prime}\\
(\sigma_{zz})_2 &= \frac{2(c_{11}-c_{12})c_{13}}{c_{11}+c_{12}}A_2^{(1)} - \left[\frac{2c_{13}^2}{c_{11}+c_{12}} - c_{33}\right]C_0^{(1)\prime} - \left[\frac{2c_{13}e_{31}}{c_{11}+c_{12}} - e_{33}\right]D_0^{(1)\prime}\\
(\sigma_{z\theta})_2 &= -c_{44}[A_0^{(1)\prime} + C_2^{(1)}]\sin\theta - e_{15}D_2^{(1)}\sin\theta\\
(\sigma_{zr})_2 &= c_{44}[A_0^{(1)\prime} + C_2^{(1)}]\cos\theta + e_{15}D_2^{(1)}\cos\theta\\
(\sigma_{r\theta})_2 &= -(c_{11}-c_{12})A_2^{(1)}\sin 2\theta,
\end{aligned} \tag{50}$$

$$\begin{aligned}
(D_r)_2 &= [A_0^{(1)\prime} + C_2^{(1)}]e_{15}\cos\theta - \epsilon_{11}D_2^{(1)}\cos\theta\\
(D_\theta)_2 &= -[A_0^{(1)\prime} + C_2^{(1)}]e_{15}\sin\theta + \epsilon_{11}D_2^{(1)}\sin\theta\\
(D_z)_2 &= \frac{2(c_{11}-c_{12})e_{31}}{c_{11}+c_{12}}A_2^{(1)} + \left[\frac{2c_{13}}{c_{11}+c_{12}}e_{31} + e_{33}\right]C_0^{(1)\prime} - \left[\frac{2e_{31}^2}{c_{11}+c_{12}} + \epsilon_{33}\right]D_0^{(1)\prime}.
\end{aligned} \tag{51}$$

It is clear that the above procedure can be continued for values of $n \geqslant 3$. The main difference will reside in the existence of non-vanishing terms in the right-hand sides of eqns (26.1)–(26.4). Therefore, particular solutions will have to be added to eqns (35) and (37). The corresponding solutions will lead to stress and electric displacement components in powers of $r^{1/2}$, r^1, $r^{3/2}$, etc., which for the sake of brevity are omitted.

Observation. If we consider the case of an elastic isotropic medium with a crack and no electric field, then the elastic constants become

$$c_{11} = c_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad c_{12} = c_{13} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad c_{44} = \frac{E}{2(1+\nu)}. \tag{52}$$

In such a case the displacement functions f_n, g_n and h_n ($n = 0, 1, 2$) reduce to those given by Hartranft and Sih (1969) leading to the same stress distribution up to a factor $E/1+\nu$ due to the different definition for the displacement field.

The equations derived so far are quite general in the sense that no specific type of mechanical or electrical loading has been imposed. In this respect we could elaborate further by considering different loading conditions which will determine the structure of the functions $A_n^{(i)}, \ldots, D_n^{(i)}$ in terms of even or odd powers of z. In particular we turn our attention to the following problem.

THE ANTIPLANE PROBLEM

We will use this problem in order to verify the validity of our results. We consider the conditions analyzed by Pak (1987), that is, the interaction of out-of-plane stresses with the in-plane (x–y plane) electric field components. For purposes of comparison we need to transform the cylindrical components of stress and electric displacement into Cartesian components. The only term of interest in the series expansion, is the one corresponding to $n = 1$. The field variables, therefore, can now be written as

$$\begin{aligned}
\sigma_{zx} &= -\frac{1}{2\sqrt{r}}[c_{44}C_1^{(2)} + e_{15}D_1^{(2)}] \sin\frac{\theta}{2} \\
\sigma_{zy} &= \frac{1}{2\sqrt{r}}[c_{44}C_1^{(2)} + e_{15}D_1^{(2)}] \cos\frac{\theta}{2} \\
D_x &= -\frac{1}{2\sqrt{r}}[e_{15}C_1^{(2)} - \epsilon_{11}D_1^{(2)}] \sin\frac{\theta}{2} \\
D_y &= \frac{1}{2\sqrt{r}}[e_{15}C_1^{(2)} - \epsilon_{11}D_1^{(2)}] \cos\frac{\theta}{2},
\end{aligned} \tag{53}$$

while the electric potential, the electric field, and the elastic displacement become

$$\begin{aligned}
\phi &= \sqrt{r}D_1^{(2)} \sin\frac{\theta}{2} \\
E_x &= \frac{1}{2\sqrt{r}}D_1^{(2)} \sin\frac{\theta}{2} \\
E_y &= \frac{1}{2\sqrt{r}}D_1^{(2)} \cos\frac{\theta}{2} \\
u_z &= \sqrt{r}C_1^{(2)} \sin\frac{\theta}{2}.
\end{aligned} \tag{54}$$

It should be noted that since in this case the independent variables are x and y only, the coefficients $C_1^{(2)}$ and $D_1^{(2)}$ entering in the above expressions are independent of z. Furthermore, if we redefine these constants according to

$$\tfrac{1}{2}C_1^{(2)} = K^S, \quad \tfrac{1}{2}D_1^{(2)} = -K^E, \tag{55}$$

we recover the results obtained in the aforementioned reference by a completely different method. The constants K^S and K^E were defined by Pak (1987) as the "strain" and "electric field" intensity factors, respectively.

DISCUSSION

A three-dimensional analysis of an infinite piezoelectric medium containing a crack has been performed. The behavior of the stress and electric field components in the neighborhood of the crack tip was obtained by means of the method of eigenfunction expansions showing the classical $1/\sqrt{r}$ type of singularity.

If we consider the x–y plane to be the in-plane surface and the z-axis to be the out-of-plane direction, as has been done in this analysis, the constitutive equation (A1) shows that the out-of-plane electric field (E_z) becomes coupled with the in-plane stresses (σ_{xx}, σ_{yy}), while the in-plane electric fields (E_x, E_y) become coupled with the out-of-plane stresses (σ_{zx}, σ_{zy}). However, the results of this analysis also show that at the crack tip ($n = 1$) only the in-plane electric field and the out-of-plane stresses become coupled, indicating that the in-plane electric field strongly influences the out-of-plane crack tip stresses and vice versa. The results also demonstrate that the interaction between the out-of-plane electric field and the in-plane stresses can be observed for the value of $n \geqslant 2$, which indicates that the out-of-plane electric field does not become singular and does not induce singular stresses at the crack tip. This becomes clear in light of fact that the out-of-plane electric field is parallel to the crack faces. Thus E_z does not get disturbed by the crack. It is demonstrated here that the coupling phenomenon at the crack tip is strongly influenced by the crack orientation with respect to the material anisotropy.

It is also important to note that thus far the analysis has led to qualitative results in the sense that we know the radial and angular behavior of the leading terms of the stress and electric field components. However, the amplitudes of the coefficient functions in terms of z remains unknown, and their determination is constrained to the particular boundary value crack problem under investigation.

We also emphasize that this analysis was three-dimensional in the sense that three independent coordinates were employed to characterize the behavior of the fields surrounding the crack. Our study can also be extended to the case in which the medium is a piezoelectric thick plate containing a through crack. However, it is important to realize that the results in such a case could be valid, at best, only in regions interior to the plate. In its present form, our formulation cannot address thickness effects.

Finally, we note that for the material symmetry and crack orientation just considered, the in-plane problem would not provide useful information because no interaction exists between in-plane stresses and in-plane electric field. Because the solution to an in-plane coupling problem is of vital importance in understanding the influence of electro-elastic coupling on the crack propagation, the authors are currently working on a plane strain problem in which the leading edge of an embedded crack is along an axis other than the axis of anisotropy.

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REFERENCES

Deeg, W. F. (1980). The analysis of dislocation, crack, and inclusion problems in piezoelectric solids. Ph.D. Thesis, Stanford University.

Frieman, S. W. (1986). Mechanical behavior of ferroelectric ceramics. *IEEE Sixth International Symp. on Application of Ferroelectrics*, pp. 367–373.

Harrison, W. B., McHenry, K. D. and Koepke, B. G. (1986). Monolithic multilayer piezoelectric ceramic transducers. *IEEE Sixth International Symp. on Application of Ferroelectrics*, pp. 265–272.

Hartranft, R. J. and Sih, G. C. (1969). The use of eigenfunction expansions in the general solution of three-dimensional crack problems. *J. Math. Mech.* **19**, 123–138.
Pak, Y. E. (1987). Crack extension force in a piezoelectric material. Advanced Development Report 20-11-87.1, Grumman Corporation.
Parton, V. Z. (1976). Fracture mechanics of piezoelectric materials. *Acta Astronautica* **3**, 671–683.
Pisarenko, G. G., Chushko, V. M. and Kovalev, S. P. (1985). Anisotropy of fracture toughness of piezoelectric ceramics. *J. Am. Ceram. Soc.* **68**, 259–265.
Tiersten, H. F. (1969). *Linear Piezoelectric Plate Vibrations*. Plenum Press, New York.

APPENDIX

The constitutive relations describing a transversely isotropic piezoelectric material with z being the anisotropic axis are given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{zy} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} \tag{A1}$$

$$\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{zy} \\ 3\varepsilon_{zx} \\ 2\varepsilon xy \end{Bmatrix} + \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{11} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} \tag{A2}$$